Numerical Algorithms

x	0	1	2	3	4	5	6	7	8	9
x^{-1}		1		7				3		9

Outline

- Divisibility and primes
- Modular arithmetic
- Euclid's GCD algorithm
- Multiplicative inverses
- Powers
- Fermat's little theorem
- Euler's theorem

Facts About Numbers

- Prime number p:
  - p is an integer
  - $p \geq 2$
  - The only divisors of p are 1and p
- Examples
  - 2, 7, 19 are primes
  - −3, 1, 6 are not primes
- Prime decomposition of a positive integer n:

$$n = p_1^{e_1} \times \ldots \times p_k^{e_k}$$

- Example:
  - $200 = 2^3 \times 5^2$

Fundamental Theorem of Arithmetic

The prime decomposition of a positive integer is unique

Greatest Common Divisor

- The greatest common divisor (GCD) of two positive integers a and b, denoted $\gcd(a, b)$, is the largest positive integer that divides both a and b
- The above definition is extended to arbitrary integers
- Examples:

$\gcd(18, 30) = 6$ $\quad$ $\gcd(0, 20) = 20$

$\gcd(-21, 49) = 7$

- Two integers a and b are said to be relatively prime if

$$\gcd(a, b) = 1$$

- Example:
  - Integers 15 and 28 are relatively prime

Modular Arithmetic

- Modulo operator for a positive integer n

$$r = a \bmod n$$

equivalent to

$$a = r + kn$$

and

$$r = a - \lfloor a/n \rfloor n$$

- Example:

$29 \bmod 13 = 3$ $\quad$ $13 \bmod 13 = 0$ $\quad$ $-1 \bmod 13 = 12$

$29 = 3 + 2 \times 13$ $\quad$ $13 = 0 + 1 \times 13$ $\quad$ $12 = -1 + 1 \times 13$

- Modulo and GCD:

$$\gcd(a, b) = \gcd(b, a \bmod b)$$

- Example:

$\gcd(21, 12) = 3$ $\quad$ $\gcd(12, 21 \bmod 12) = \gcd(6, 9) = 3$

Euclid's GCD Algorithm

- Euclid's algorithm for computing the GCD repeatedly applies the formula

$$\gcd(a, b) = \gcd(b, a \bmod b)$$

- Example
  - $\gcd(412, 260) = 4$

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Algorithm EuclidGCD(a, b)
  Input integers a and b
  Output gcd(a, b)

  if b = 0
    return a
  else
    return EuclidGCD(b, a mod b)
```

a	412	260	152	108	44	20	4
b	260	152	108	44	20	4	0

Analysis

- Let a_i and b_i be the arguments of the i-th recursive call of algorithm *EuclidGCD*
- We have

  $a_{i+2} = b_{i+1} = a_i \bmod a_{i+1} < a_{i+1}$
- Sequence $a_1, a_2, \ldots, a_n$ decreases exponentially, namely

  $a_{i+2} \le \frac{1}{2} a_i$ for $i > 1$

  Case 1 $a_{i+1} \le \frac{1}{2} a_i$ $\quad a_{i+2} < a_{i+1} \le \frac{1}{2} a_i$

  Case 2 $a_{i+1} > \frac{1}{2} a_i$ $\quad a_{i+2} = a_i \bmod a_{i+1} = a_i - a_{i+1} \le \frac{1}{2} a_i$
- Thus, the maximum number of recursive calls of algorithm *EuclidGCD*(a, b) is

  $1 + 2 \log \max(a, b)$
- Algorithm *EuclidGCD*(a, b) executes $O(\log \max(a, b))$ arithmetic operations

Multiplicative Inverses (1)

- The residues modulo a positive integer n are the set

  $Z_n = \{0, 1, 2, \ldots, (n-1)\}$
- Let x and y be two elements of Z_n such that

  $xy \bmod n = 1$

  We say that y is the multiplicative inverse of x in Z_n and we write $y = x^{-1}$
- Example:
  - Multiplicative inverses of the residues modulo 11

x	0	1	2	3	4	5	6	7	8	9	10
x^{-1}		1	6	4	3	9	2	8	7	5	10

Multiplicative Inverses (2)

Theorem

An element x of Z_n has a multiplicative inverse if and only if x and n are relatively prime

- Example
  - The elements of Z_{10} with a multiplicative inverse are $1, 3, 5, 7$

Corollary

If is p is prime, every nonzero residue in Z_p has a multiplicative inverse

Theorem

A variation of Euclid's GCD algorithm computes the multiplicative inverse of an element x of Z_n or determines that it does not exist

x	0	1	2	3	4	5	6	7	8	9
x^{-1}		1		7				3		9

Powers

- Let p be a prime
- The sequences of successive powers of the elements of Z_p exhibit repeating subsequences
- The sizes of the repeating subsequences and the number of their repetitions are the divisors of $p - 1$
- Example ($p = 7$)

x	x^2	x^3	x^4	x^5	x^6
1	1	1	1	1	1
2	4	1	2	4	1
3	2	6	4	5	1
4	2	1	4	2	1
5	4	6	2	3	1
6	1	6	1	6	1

Fermat's Little Theorem

Theorem

Let p be a prime. For each nonzero residue x of Z_p, we have $x^{p-1} \bmod p = 1$

- Example ($p = 5$):

  $1^4 \bmod 5 = 1$ $\quad 2^4 \bmod 1 = 16 \bmod 5 = 1$

  $3^4 \bmod 1 = 81 \bmod 5 = 1$ $\quad 4^4 \bmod 1 = 256 \bmod 5 = 1$

Corollary

Let p be a prime. For each nonzero residue x of Z_p, the multiplicative inverse of x is $x^{p-2} \bmod p$

Proof

$x(x^{p-2} \bmod p) \bmod p = xx^{p-2} \bmod p = x^{p-1} \bmod p = 1$

Euler's Theorem

- The multiplicative group for Z_n, denoted with Z^*_n, is the subset of elements of Z_n relatively prime with n
- The totient function of n, denoted with $\phi(n)$, is the size of Z^*_n
- Example

  $Z^*_{10} = \{1, 3, 7, 9\}$ $\quad \phi(10) = 4$
- If p is prime, we have

  $Z^*_p = \{1, 2, \ldots, (p-1)\}$ $\quad \phi(p) = p - 1$

Theorem

For each element x of Z^*_n, we have $x^{\phi(n)} \bmod n = 1$

- Example ($n = 10$)

  $3^{\phi(10)} \bmod 10 = 3^4 \bmod 10 = 81 \bmod 10 = 1$

  $7^{\phi(10)} \bmod 10 = 7^4 \bmod 10 = 2401 \bmod 10 = 1$

  $9^{\phi(10)} \bmod 10 = 9^4 \bmod 10 = 6561 \bmod 10 = 1$