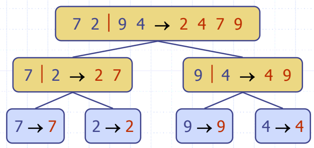
Merge Sort



Outline and Reading

- Divide-and-conquer paradigm (§4.1.1)
- Merge-sort (§4.1.1)
  - Algorithm
  - Merging two sorted sequences
  - Merge-sort tree
  - Execution example
  - Analysis
- Generic merging and set operations (§4.2.1)
- Summary of sorting algorithms (§4.2.1)

Divide-and-Conquer

- Divide-and conquer is a general algorithm design paradigm:
  - Divide: divide the input data S in two disjoint subsets S_1 and S_2
  - Recur: solve the subproblems associated with S_1 and S_2
  - Conquer: combine the solutions for S_1 and S_2 into a solution for S
- The base case for the recursion are subproblems of size 0 or 1
- Merge-sort is a sorting algorithm based on the divide-and-conquer paradigm
- Like heap-sort
  - It uses a comparator
  - It has $O(n \log n)$ running time
- Unlike heap-sort
  - It does not use an auxiliary priority queue
  - It accesses data in a sequential manner (suitable to sort data on a disk)

Merge-Sort

- Merge-sort on an input sequence S with n elements consists of three steps:
  - Divide: partition S into two sequences S_1 and S_2 of about $n/2$ elements each
  - Recur: recursively sort S_1 and S_2
  - Conquer: merge S_1 and S_2 into a unique sorted sequence

Algorithm *mergeSort(S, C)*
  Input sequence S with n elements, comparator C
  Output sequence S sorted according to C
  if $S.size() > 1$
    $(S_1, S_2) \leftarrow partition(S, n/2)$
    $mergeSort(S_1, C)$
    $mergeSort(S_2, C)$
    $S \leftarrow merge(S_1, S_2)$

Merging Two Sorted Sequences

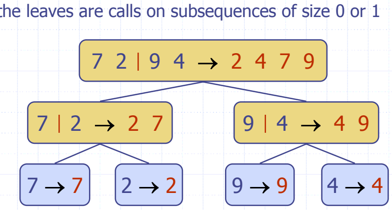
- The conquer step of merge-sort consists of merging two sorted sequences A and B into a sorted sequence S containing the union of the elements of A and B
- Merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes $O(n)$ time

Algorithm *merge(A, B)*
  Input sequences A and B with $n/2$ elements each
  Output sorted sequence of $A \cup B$

  $S \leftarrow$ empty sequence
  while $\neg A.isEmpty() \wedge \neg B.isEmpty()$
    if $A.first().element() < B.first().element()$
      $S.insertLast(A.remove(A.first()))$
    else
      $S.insertLast(B.remove(B.first()))$
  while $\neg A.isEmpty()$
  $S.insertLast(A.remove(A.first()))$
  while $\neg B.isEmpty()$
  $S.insertLast(B.remove(B.first()))$
  return S

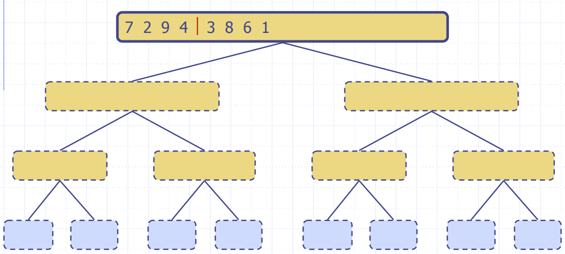
Merge-Sort Tree

- An execution of merge-sort is depicted by a binary tree
  - each node represents a recursive call of merge-sort and stores
    - unsorted sequence before the execution and its partition
    - sorted sequence at the end of the execution
  - the root is the initial call
  - the leaves are calls on subsequences of size 0 or 1



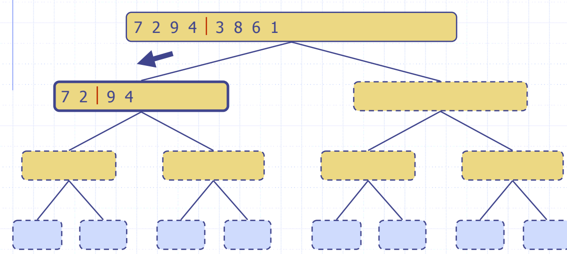
Execution Example

- Partition



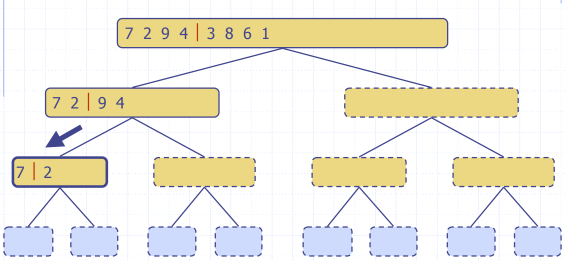
Execution Example (cont.)

- Recursive call, partition



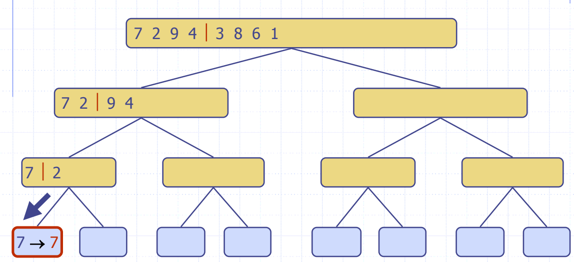
Execution Example (cont.)

- Recursive call, partition



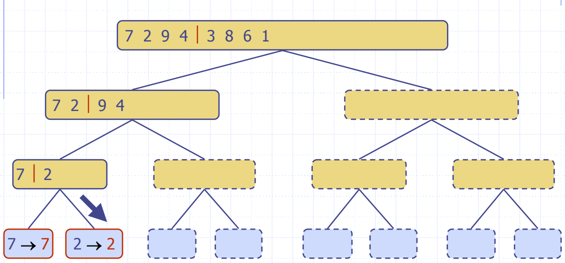
Execution Example (cont.)

- Recursive call, base case



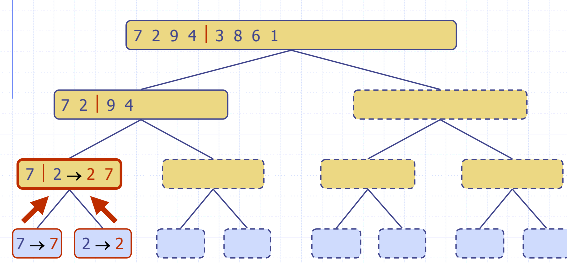
Execution Example (cont.)

- Recursive call, base case



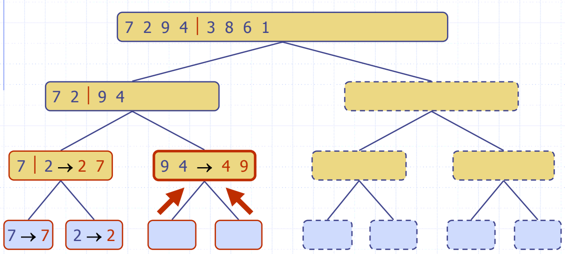
Execution Example (cont.)

- Merge



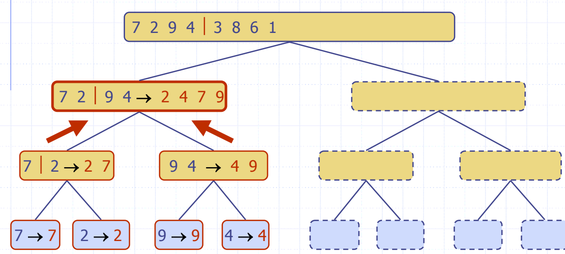
Execution Example (cont.)

- Recursive call, …, base case, merge



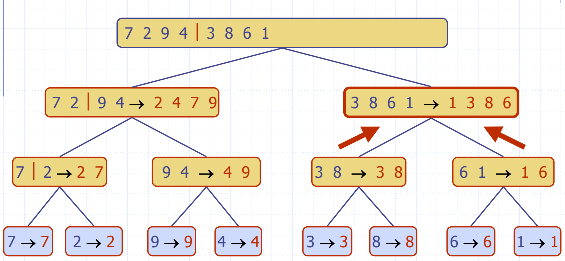
Execution Example (cont.)

- Merge



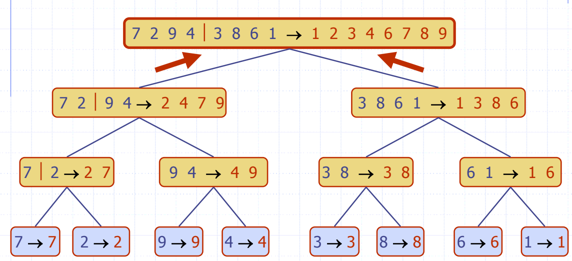
Execution Example (cont.)

- Recursive call, …, merge, merge



Execution Example (cont.)

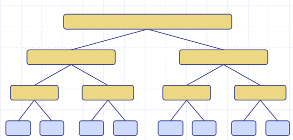
- Merge



Analysis of Merge-Sort

- The height h of the merge-sort tree is $O(\log n)$
  - at each recursive call we divide in half the sequence,
- The overall amount or work done at the nodes of depth i is $O(n)$
  - we partition and merge 2^i sequences of size $n/2^i$
  - we make 2^{i+1} recursive calls
- Thus, the total running time of merge-sort is $O(n \log n)$

depth	#seqs	size
0	1	n
1	2	$n/2$
i	2^i	$n/2^i$
…	…	…



Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	slow; in-place; for small data sets (< 1K)
insertion-sort	$O(n^2)$	slow; in-place; for small data sets (< 1K)
heap-sort	$O(n \log n)$	fast; in-place; for large data sets (1K — 1M)
merge-sort	$O(n \log n)$	fast; sequential data access; for huge data sets (> 1M)