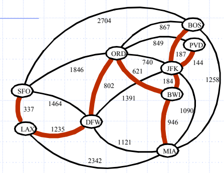
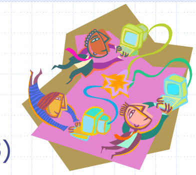
Minimum Spanning Trees



Outline and Reading



- Minimum Spanning Trees (§7.3)
  - Definitions
  - A crucial fact
- The Prim-Jarnik Algorithm (§7.3.2)
- Kruskal's Algorithm (§7.3.1)
- Baruvka's Algorithm (§7.3.3)

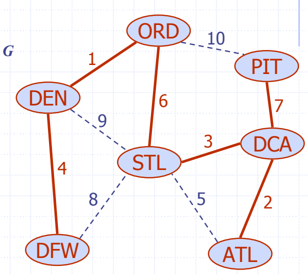
Minimum Spanning Tree

Spanning subgraph
- Subgraph of a graph G containing all the vertices of G

Spanning tree
- Spanning subgraph that is itself a (free) tree

Minimum spanning tree (MST)
- Spanning tree of a weighted graph with minimum total edge weight

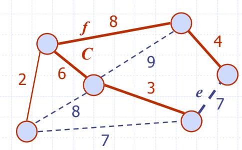
- Applications
  - Communications networks
  - Transportation networks



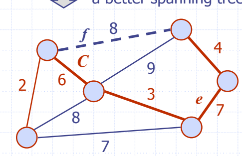
Cycle Property

Cycle Property:
- Let T be a minimum spanning tree of a weighted graph G
- Let e be an edge of G that is not in T and C let be the cycle formed by e with T
- For every edge f of C, $weight(f) \leq weight(e)$

Proof:
- By contradiction
- If $weight(f) > weight(e)$ we can get a spanning tree of smaller weight by replacing e with f



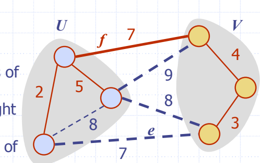
Replacing f with e yields a better spanning tree



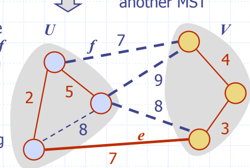
Partition Property

Partition Property:
- Consider a partition of the vertices of G into subsets U and V
- Let e be an edge of minimum weight across the partition
- There is a minimum spanning tree of G containing edge e

Proof:
- Let T be an MST of G
- If T does not contain e, consider the cycle C formed by e with T and let f be an edge of C across the partition
- By the cycle property, $weight(f) \leq weight(e)$
- Thus, $weight(f) = weight(e)$
- We obtain another MST by replacing f with e

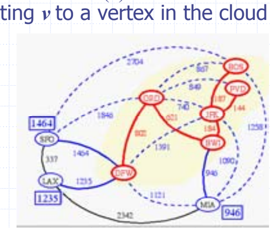


Replacing f with e yields another MST



Prim-Jarnik's Algorithm

- Similar to Dijkstra's algorithm (for a connected graph)
- We pick an arbitrary vertex s and we grow the MST as a cloud of vertices, starting from s
- We store with each vertex v a label $d(v)$ = the smallest weight of an edge connecting v to a vertex in the cloud
- At each step:
  - We add to the cloud the vertex u outside the cloud with the smallest distance label
  - We update the labels of the vertices adjacent to u

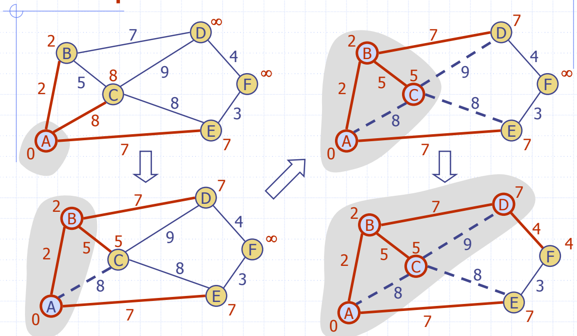


Prim-Jarnik's Algorithm (cont.)

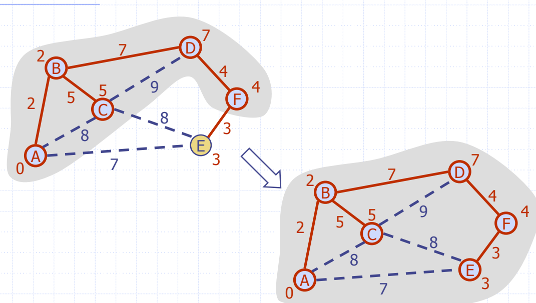
- A priority queue stores the vertices outside the cloud
  - Key: distance
  - Element: vertex
- Locator-based methods
  - *insert*(*k*,*e*) returns a locator
  - *replaceKey*(*l*,*k*) changes the key of an item
- We store three labels with each vertex:
  - Distance
  - Parent edge in MST
  - Locator in priority queue

```
Algorithm PrimJarnikMST(G)
  Q ← new heap-based priority queue
  s ← a vertex of G
  for all v ∈ G.vertices()
    if v = s
      setDistance(v, 0)
    else
      setDistance(v, ∞)
    setParent(v, ∅)
    l ← Q.insert(getDistance(v), v)
    setLocator(v,l)
  while ¬Q.isEmpty()
    u ← Q.removeMin()
    for all e ∈ G.incidentEdges(u)
      z ← G.opposite(u,e)
      r ← weight(e)
      if r < getDistance(z)
        setDistance(z,r)
        setParent(z,e)
        Q.replaceKey(getLocator(z),r)
```


Example



Example (contd.)



Analysis

- Graph operations
  - Method incidentEdges is called once for each vertex
- Label operations
  - We set/get the distance, parent and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex w in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Prim-Jarnik's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$
- The running time is $O(m \log n)$ since the graph is connected

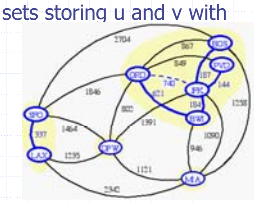
Kruskal's Algorithm

- A priority queue stores the edges outside the cloud
  - Key: weight
  - Element: edge
- At the end of the algorithm
  - We are left with one cloud that encompasses the MST
  - A tree *T* which is our MST

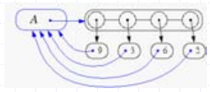
```
Algorithm KruskalMST(G)
  for each vertex V in G do
    define a Cloud(v) of ← {v}
  let Q be a priority queue.
  Insert all edges into Q using their
  weights as the key
  T ← ∅
  while T has fewer than n-1 edges do
    edge e = T.removeMin()
    Let u, v be the endpoints of e
    if Cloud(v) ≠ Cloud(u) then
      Add edge e to T
      Merge Cloud(v) and Cloud(u)
  return T
```


Data Structure for Kruskal Algortihm

- The algorithm maintains a forest of trees
- An edge is accepted it if connects distinct trees
- We need a data structure that maintains a **partition**, i.e., a collection of disjoint sets, with the operations:
  - -**find**(u): return the set storing u
  - -**union**(u,v): replace the sets storing u and v with their union



Representation of a Partition



- Each set is stored in a sequence
- Each element has a reference back to the set
  - operation find(u) takes O(1) time, and returns the set of which u is a member.
  - in operation union(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
  - the time for operation union(u,v) is $\min(n_u, n_v)$, where n_u and n_v are the sizes of the sets storing u and v
- Whenever an element is processed, it goes into a set of size at least double, hence each element is processed at most log n times

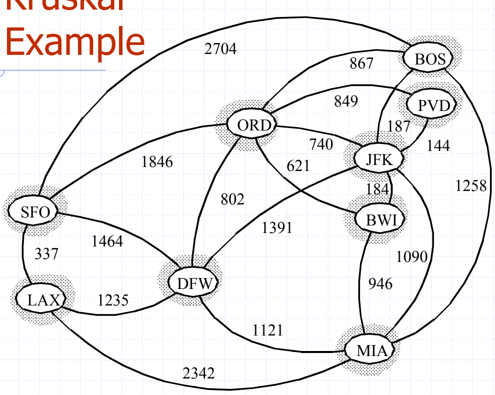
Partition-Based Implementation

- A partition-based version of Kruskal's Algorithm performs cloud merges as unions and tests as finds.

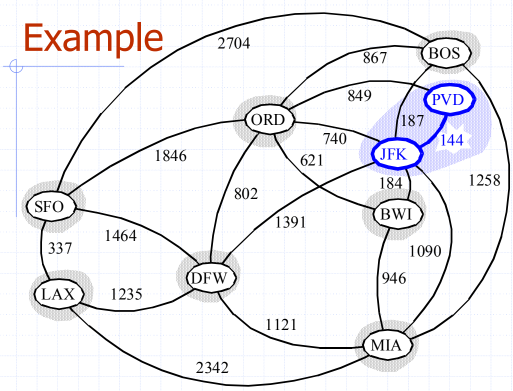
Algorithm Kruskal(*G*):
  Input: A weighted graph *G*.
  Output: An MST *T* for *G*.
Let *P* be a partition of the vertices of *G*, where each vertex forms a separate set.
Let *Q* be a priority queue storing the edges of *G*, sorted by their weights
Let *T* be an initially-empty tree
while *Q* is not empty **do**
  (*u*,*v*) ← *Q*.removeMinElement()
  if *P*.find(*u*) != *P*.find(*v*) **then**
    Add (*u*,*v*) to *T*
    P.union(*u*,*v*)
return *T*

Running time: O((n+m)log n)

Kruskal Example



Example



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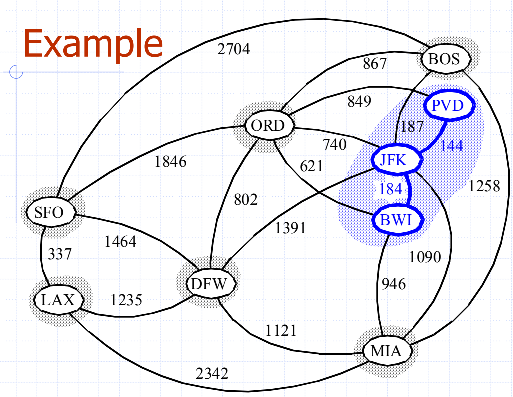

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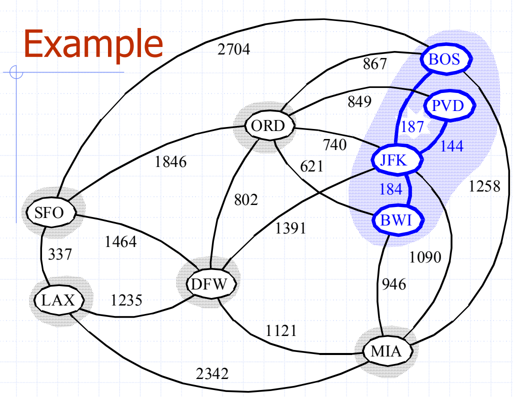

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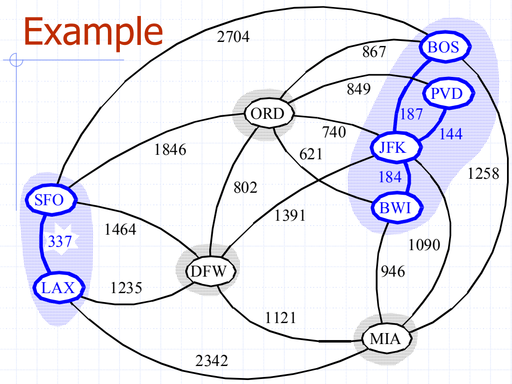

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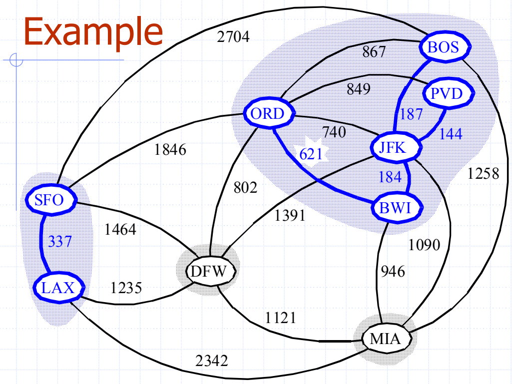

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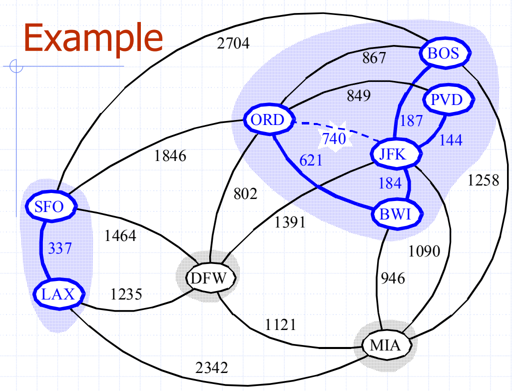

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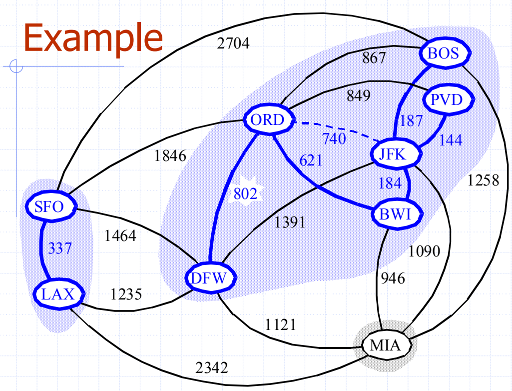
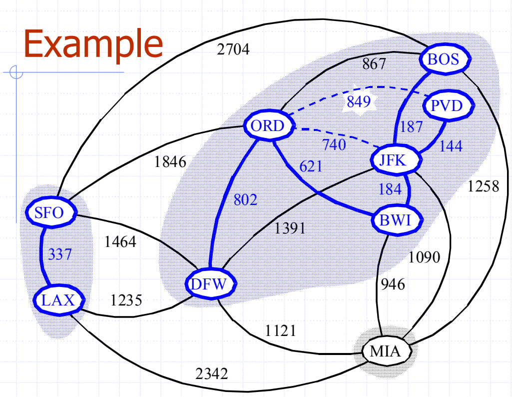


Example



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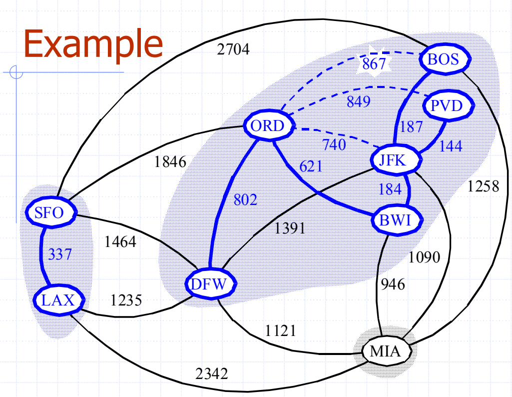

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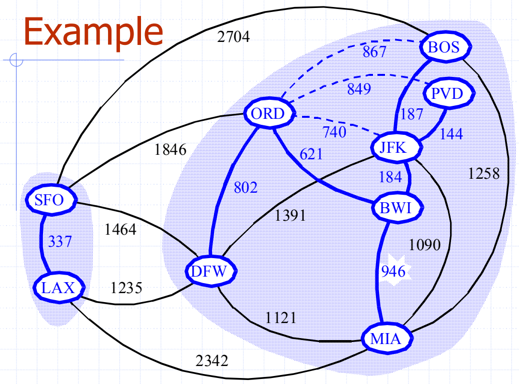

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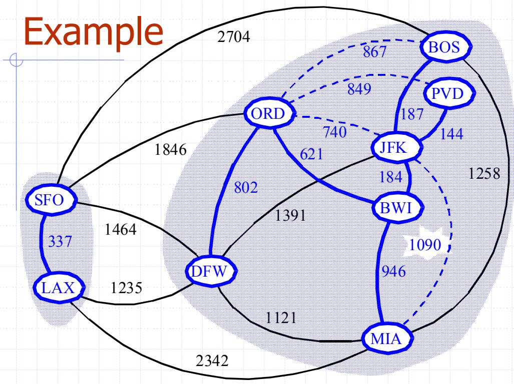

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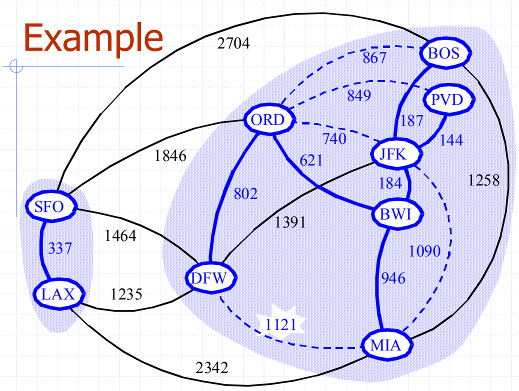

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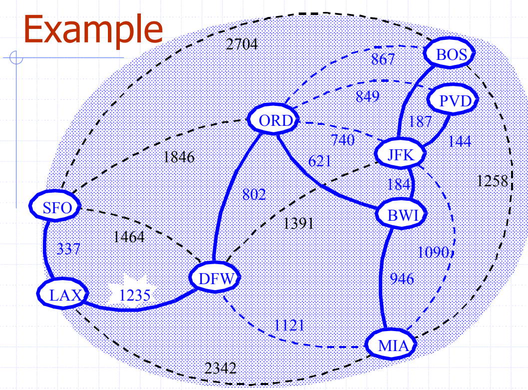


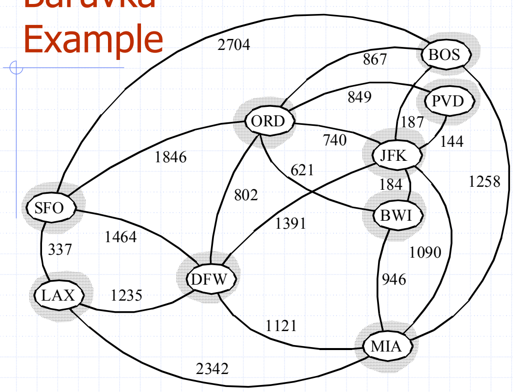
Baruvka's Algorithm

- Like Kruskal's Algorithm, Baruvka's algorithm grows many "clouds" at once.

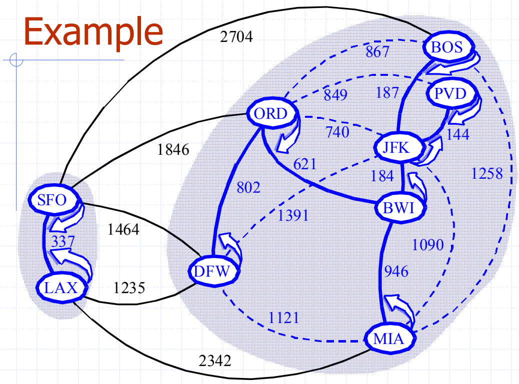
Algorithm *BaruvkaMST(G)*
 T ← *V* {just the vertices of *G*}
 while *T* has fewer than *n*-1 edges **do**
  for each connected component *C* in *T* **do**
   Let edge *e* be the smallest-weight edge from *C* to another component in *T*.
   if *e* is not already in *T* **then**
    Add edge *e* to *T*
 return *T*

- Each iteration of the while-loop halves the number of connected compontents in T.
  - The running time is O(m log n).

Baruvka Example



Example



Example