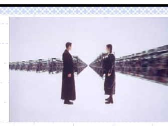
Dynamic Programming

Outline and Reading

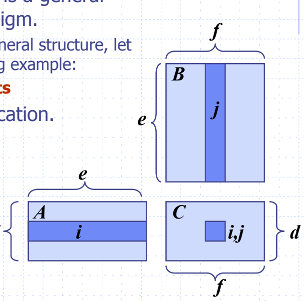
- Matrix Chain-Product (§5.3.1)
- The General Technique (§5.3.2)
- 0-1 Knapsack Problem (§5.3.3)

Matrix Chain-Products



- Dynamic Programming is a general algorithm design paradigm.
  - Rather than give the general structure, let us first give a motivating example:
  - **Matrix Chain-Products**
- Review: Matrix Multiplication.
  - $C = A*B$
  - A is $d \times e$ and B is $e \times f$
  - $O(d \cdot e \cdot f)$ time

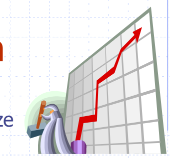
$$C[i,j] = \sum_{k=0}^{e-1} A[i,k] * B[k,j]$$



Matrix Chain-Products

- **Matrix Chain-Product:**
  - Compute $A=A_0*A_1*\ldots*A_{n-1}$
  - A_i is $d_i \times d_{i+1}$
  - Problem: How to parenthesize?
- Example
  - B is 3 × 100
  - C is 100 × 5
  - D is 5 × 5
  - (B*C)*D takes 1500 + 75 = 1575 ops
  - B*(C*D) takes 1500 + 2500 = 4000 ops

Enumeration Approach



- **Matrix Chain-Product Alg.:**
  - Try all possible ways to parenthesize $A=A_0*A_1*\ldots*A_{n-1}$
  - Calculate number of ops for each one
  - Pick the one that is best
- Running time:
  - The number of parenthesizations is equal to the number of binary trees with n nodes
  - This is **exponential**!
  - It is called the Catalan number, and it is almost 4^n.
  - This is a terrible algorithm!

Greedy Approach

- Idea #1: repeatedly select the product that uses (up) the most operations.
- Counter-example:
  - A is 10 × 5
  - B is 5 × 10
  - C is 10 × 5
  - D is 5 × 10
  - Greedy idea #1 gives (A*B)*(C*D), which takes 500+1000+500 = 2000 ops
  - A*((B*C)*D) takes 500+250+250 = 1000 ops

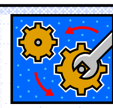
Another Greedy Approach

- Idea #2: repeatedly select the product that uses the fewest operations.
- Counter-example:
  - A is 101 × 11
  - B is 11 × 9
  - C is 9 × 100
  - D is 100 × 99
  - Greedy idea #2 gives A*((B*C)*D)), which takes 109989+9900+108900=228789 ops
  - (A*B)*(C*D) takes 9999+89991+89100=189090 ops
- The greedy approach is not giving us the optimal value.

"Recursive" Approach

- Define **subproblems**:
  - Find the best parenthesization of $A_i*A_{i+1}*\dots*A_j$.
  - Let $N_{i,j}$ denote the number of operations done by this subproblem.
  - The optimal solution for the whole problem is $N_{0,n-1}$.
- **Subproblem optimality**: The optimal solution can be defined in terms of optimal subproblems
  - There has to be a final multiplication (root of the expression tree) for the optimal solution.
  - Say, the final multiply is at index i: $(A_0*\dots*A_i)*(A_{i+1}*\dots*A_{n-1})$.
  - Then the optimal solution $N_{0,n-1}$ is the sum of two optimal subproblems, $N_{0,i}$ and $N_{i+1,n-1}$ plus the time for the last multiply.
  - If the global optimum did not have these optimal subproblems, we could define an even better "optimal" solution.

Characterizing Equation

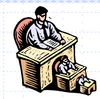


- The global optimal has to be defined in terms of optimal subproblems, depending on where the final multiply is at.
- Let us consider all possible places for that final multiply:
  - Recall that A_i is a $d_i \times d_{i+1}$ dimensional matrix.
  - So, a characterizing equation for $N_{i,j}$ is the following:

$$N_{i,j} = \min_{i \le k < j} \{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$

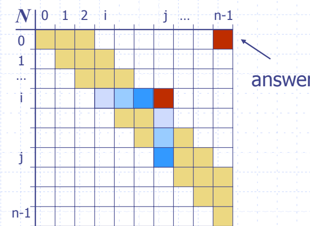
- Note that subproblems are not independent–the **subproblems overlap**.

Dynamic Programming Algorithm Visualization

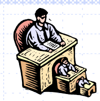


- The bottom-up construction fills in the N array by diagonals
- $N_{i,j}$ gets values from previous entries in i-th row and j-th column
- Filling in each entry in the N table takes O(n) time.
- Total run time: $O(n^3)$
- Getting actual parenthesization can be done by remembering "k" for each N entry

$$N_{i,j} = \min_{i \le k < j} \{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$



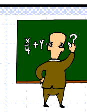
Dynamic Programming Algorithm



- Since subproblems overlap, we don't use recursion.
- Instead, we construct optimal subproblems "bottom-up."
- $N_{i,i}$'s are easy, so start with them
- Then do problems of "length" 2,3,… subproblems, and so on.
- Running time: $O(n^3)$

Algorithm *matrixChain*(*S*):
 Input: sequence S of n matrices to be multiplied
 Output: number of operations in an optimal parenthesization of S
 for $i \leftarrow 1$ **to** $n-1$ **do**
  $N_{i,i} \leftarrow 0$
 for $b \leftarrow 1$ **to** $n-1$ **do**
  { $b = j - i$ is the length of the problem }
  for $i \leftarrow 0$ **to** $n-b-1$ **do**
   $j \leftarrow i + b$
   $N_{i,j} \leftarrow +\infty$
   for $k \leftarrow i$ **to** $j-1$ **do**
    $N_{i,j} \leftarrow \min\{N_{i,j}, N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$
 return $N_{0,n-1}$

The General Dynamic Programming Technique



- Applies to a problem that at first seems to require a lot of time (possibly exponential), provided we have:
  - **Simple subproblems:** the subproblems can be defined in terms of a few variables, such as j, k, l, m, and so on.
  - **Subproblem optimality:** the global optimum value can be defined in terms of optimal subproblems
  - **Subproblem overlap:** the subproblems are not independent, but instead they overlap (hence, should be constructed bottom-up).

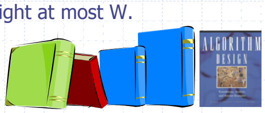
The 0/1 Knapsack Problem

- Given: A set S of n items, with each item i having
  - w_i - a positive weight
  - b_i - a positive benefit
- Goal: Choose items with maximum total benefit but with weight at most W.
- If we are **not** allowed to take fractional amounts, then this is the **0/1 knapsack problem**.
  - In this case, we let T denote the set of items we take
  - Objective: maximize $\sum_{i \in T} b_i$
  - Constraint: $\sum_{i \in T} w_i \le W$

Example

- Given: A set S of n items, with each item i having
  - b_i - a positive "benefit"
  - w_i - a positive "weight"
- Goal: Choose items with maximum total benefit but with weight at most W.

Items:



Items:	1	2	3	4	5
Weight:	4 in	2 in	2 in	6 in	2 in
Benefit:	\$20	\$3	\$6	\$25	\$80

"knapsack"

box of width 9 in

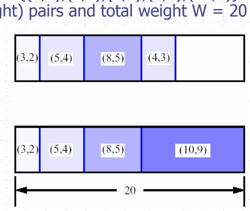
Solution:
- item 5 (\$80, 2 in)
- item 3 (\$6, 2in)
- item 1 (\$20, 4in)

A 0/1 Knapsack Algorithm, First Attempt

- S_k: Set of items numbered 1 to k.
- Define B[k] = best selection from S_k.
- Problem: does not have subproblem optimality:
  - Consider set S={(3,2),(5,4),(8,5),(4,3),(10,9)} of (benefit, weight) pairs and total weight W = 20

Best for S_4:

Best for S_5:



A 0/1 Knapsack Algorithm, Second Attempt

- S_k: Set of items numbered 1 to k.
- Define B[k,w] to be the best selection from S_k with weight at most w
- Good news: this does have subproblem optimality.

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w],\ B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

- I.e., the best subset of S_k with weight at most w is either
  - the best subset of S_{k-1} with weight at most w or
  - the best subset of S_{k-1} with weight at most $w-w_k$ plus item k

0/1 Knapsack Algorithm

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w],\ B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

- Recall the definition of B[k,w]
- Since B[k,w] is defined in terms of B[k−1,*], we can use two arrays of instead of a matrix
- Running time: O(nW).
- Not a polynomial-time algorithm since W may be large
- This is a pseudo-polynomial time algorithm

```
Algorithm 01Knapsack(S, W):
  Input: set S of n items with benefit b_i
         and weight w_i; maximum weight W
  Output: benefit of best subset of S with
         weight at most W
  let A and B be arrays of length W + 1
  for w ← 0 to W do
    B[w] ← 0
  for k ← 1 to n do
    copy array B into array A
    for w ← w_k to W do
      if A[w−w_k] + b_k > A[w] then
        B[w] ← A[w−w_k] + b_k
  return B[W]
```